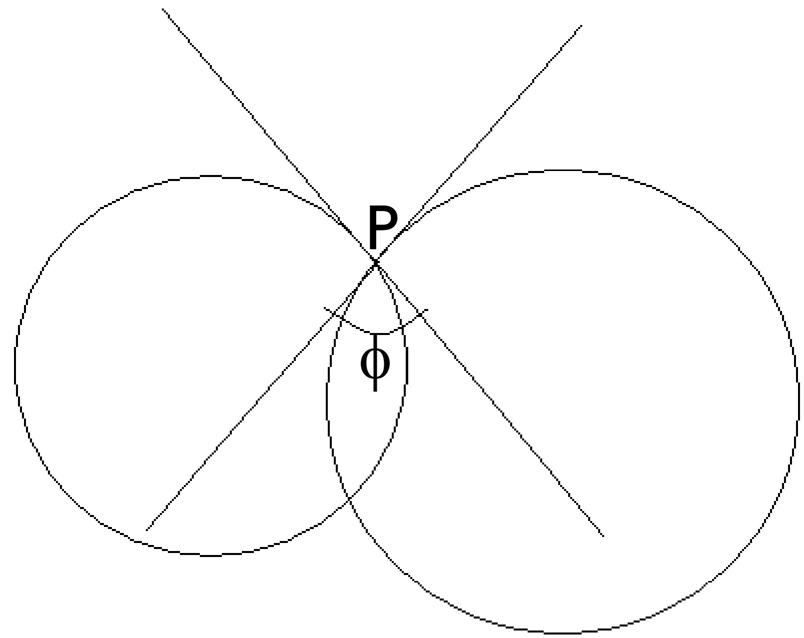
System of Circles

Angle of intersection of two circles

- The angle of intersection of two curves is measured by the angle between the tangents to the curves at a point of their intersection.



In case of circle

the radius to the point of contact of tangent

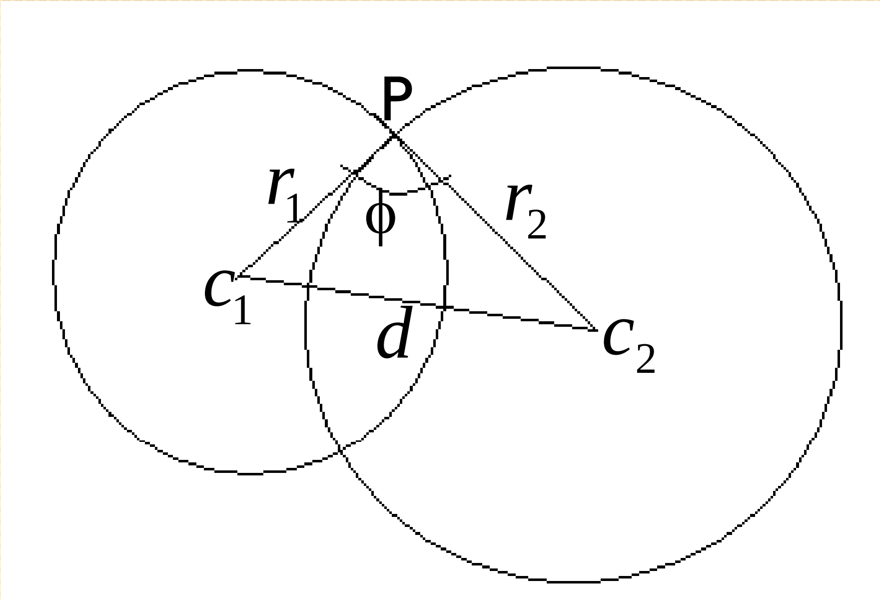
is perpendicular to the tangent.

So, the angle between two tangents

to the circles at a common point

is equal to the angle between the radii

of the circles drawn to the same point.



$$\cos\phi = \frac{r_1^{\,2} + r_2^{\,2} - d^{\,2}}{2r_1r_2}$$

r_1 = radius of left circle

r_2 = radius of right circle

c_1 = centre of left circle

c_2 = centre of right circle

ϕ = angle of intersection of the circles

d = distance between centres of the circles

- If the equation of the circle be

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$,

then $r_1^2 = g_1^2 + f_1^2 - c_1$

and $r_2^2 = g_2^2 + f_2^2 - c_2$.

- So, $d^2 = (g_1 - g_2)^2 + (f_1 - f_2)^2$

$$\therefore \cos\phi = \left(g_1{}^2 + f_1{}^2 - c_1 + g_2{}^2 + f_2{}^2 - c_2 - (g_1 - g_2)^2 - (f_1 - f_2)^2\right) / 2r_1r_2$$

$$\text{or, } \cos\phi = \left(g_1{}^2 + f_1{}^2 - c_1 + g_2{}^2 + f_2{}^2 - c_2 - g_1{}^2 - g_2{}^2 + 2g_1g_2 - f_1{}^2 - f_2{}^2 + 2f_1f_2\right) / 2r_1r_2$$

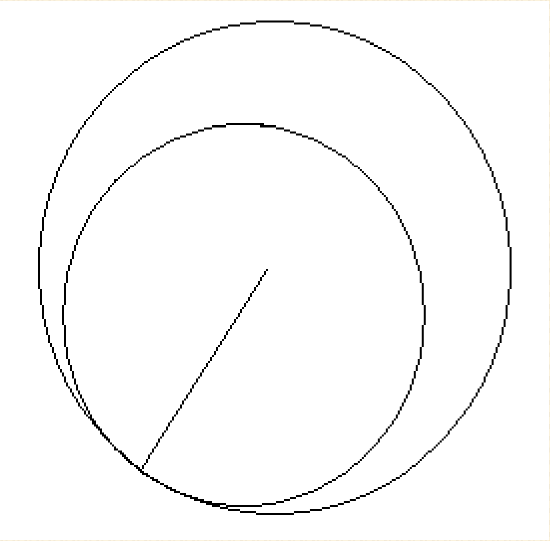
$$\cos\phi = \frac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2r_1r_2}$$

Properties

$\phi = 0°$

Two circles touch each other internally

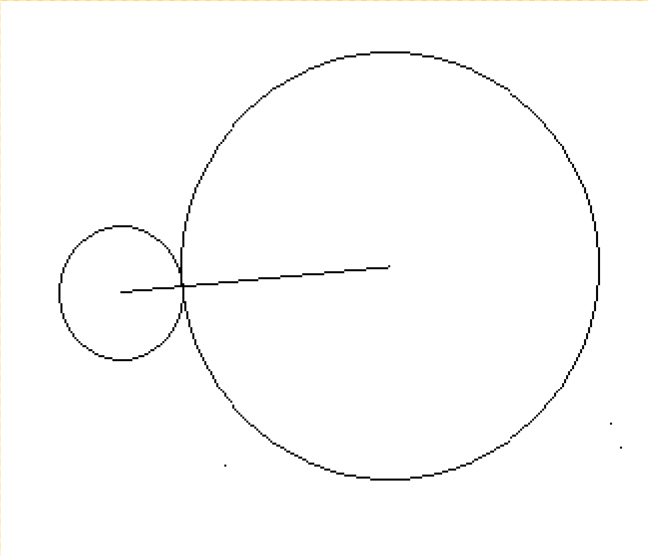
$$2g_1g_2 + 2f_1f_2 - c_1 - c_2 = 2r_1r_2$$



$\phi = 180°$

Two circles touch each other externally

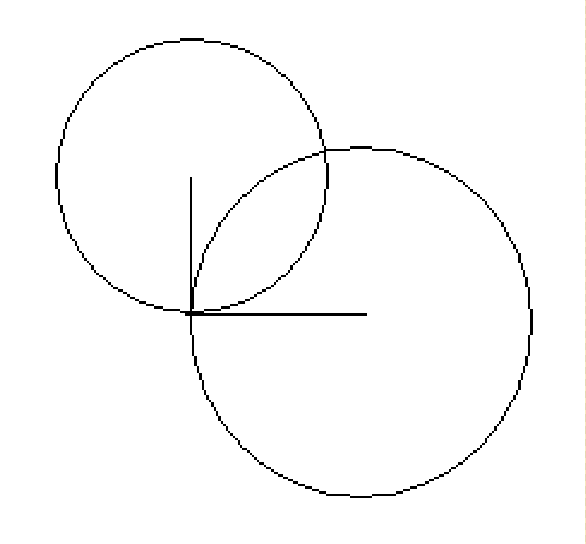
$$2g_1g_2 + 2f_1f_2 - c_1 - c_2 = -2r_1r_2$$



$\phi=90°$

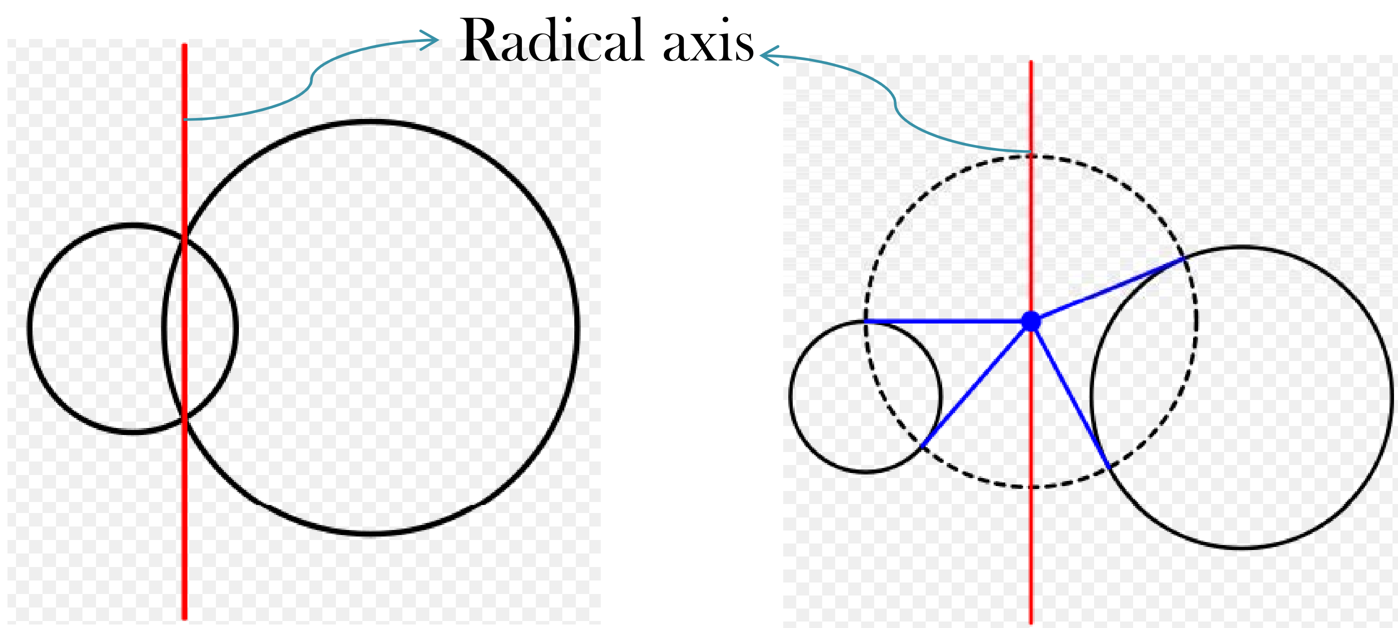
Two circles are orthogonal

$$2g_1g_2 + 2f_1f_2 - c_1 - c_2 = 0$$



Radical Axis

- The radical axis of circles is the locus of a point which moves so that the lengths of tangents drawn from it to the circles are equal.



Radical Axis of Two Circles

- Let the equation of the circles be

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \ldots \qquad \ldots (1)$$

and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \ldots \qquad \ldots (2)$

- Let (x_1, y_1) be the point from which the lengths of two tangents to eq(1) and eq(2) are equal.

$$\therefore x_1^{\,2} + y_1^{\,2} + 2g_1x_1 + 2f_1y_1 + c_1 = x_1^{\,2} + y_1^{\,2} + 2g_2x_1 + 2f_2y_1 + c_2$$

or, $2(g_1 - g_2)x_1 + 2(f_1 - f_2)y_1 + c_1 - c_2 = 0$

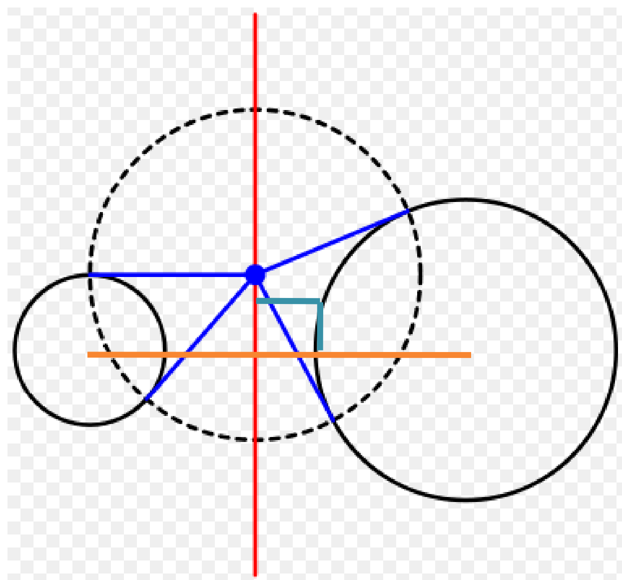
- Hence the locus of the point (x_1, y_1) is a straight line; *i.e.* the radical axis of the two circles is

$$\boxed{2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0}$$

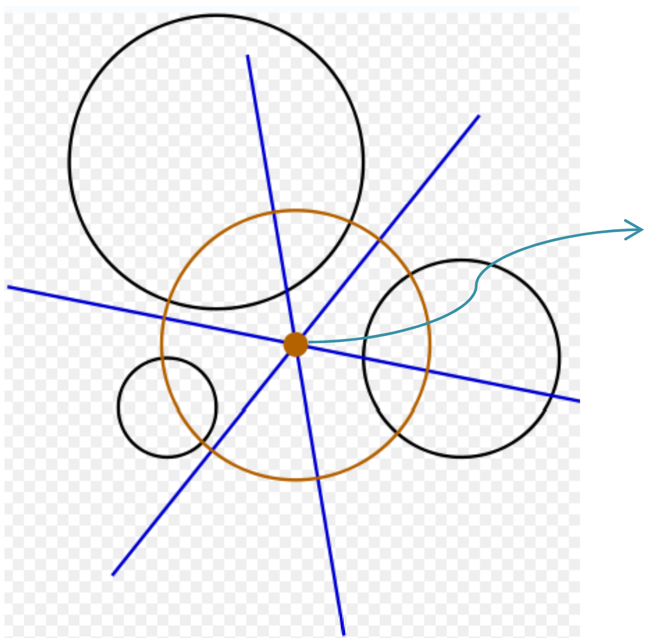
which is a straight line.

Properties of Radical Axis

- The radical axis of two circles is perpendicular to the line joining the centres.



- The radical axes of three circles, taken in pair meet in a point. This point is called radical centre.



Problems

1. Prove that the circles $x^2 + y^2 - 3x + 8y - 2 = 0$ and $x^2 + y^2 + 4x - 5y - 24 = 0$ cut orthogonally.

2. Find the angle between the circles $x^2 + y^2 - 8x - 12y = 0$ and $x^2 + y^2 - 2x + 4y = 0$.

3. Find the radical axis of the pair of circles $x^2 + y^2 = 144$ and $x^2 + y^2 - 15x + 11y = 0$.

4. Find the radical axis of the following pair of circles.

C1: $x^2 + y^2 - 3x - 4y + 5 = 0$

C2: $3x^2 + 3y^2 - 7x + 8y + 11 = 0$

5. Find the radical centre of the following three circles.

C1: $x^2 + y^2 + x + 2y + 3 = 0$

C2: $x^2 + y^2 + 2x + 4y + 5 = 0$

C3: $x^2 + y^2 - 7x - 8y - 9 = 0$

6. Find the radical centre of the following three circles.

C1: $x^2 + y^2 + 3x - 2y - 4 = 0$

C2: $x^2 + y^2 - 2x - y - 6 = 0$

C3: $x^2 + y^2 - 1 = 0$